OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE

MATHEMATICS (MEI) 4753/01

Methods for Advanced Mathematics (C3)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- Graph paper
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
None

Friday 5 June 2009
Afternoon

Duration: 1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (36 marks)

1 Evaluate $\int_0^{\frac{1}{6}\pi} \sin 3x \, dx$. **[3]**

2 A radioactive substance decays exponentially, so that its mass M grams can be modelled by the equation $M = Ae^{-kt}$, where t is the time in years, and A and k are positive constants.

(i) An initial mass of 100 grams of the substance decays to 50 grams in 1500 years. Find A and k. **[5]**

(ii) The substance becomes safe when 99% of its initial mass has decayed. Find how long it will take before the substance becomes safe. **[3]**

3 Sketch the curve $y = 2 \arccos x$ for $-1 \leqslant x \leqslant 1$. **[3]**

4 Fig. 4 shows a sketch of the graph of $y = 2|x - 1|$. It meets the x- and y-axes at $(a, 0)$ and $(0, b)$ respectively.

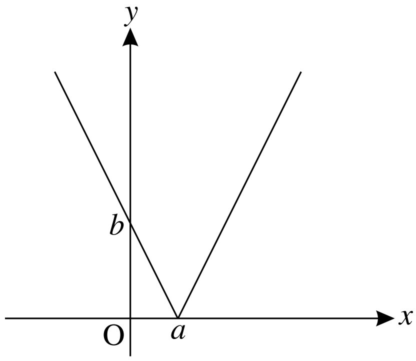


Fig. 4

Find the values of a and b. **[3]**

5 The equation of a curve is given by $e^{2y} = 1 + \sin x$.

(i) By differentiating implicitly, find $\frac{dy}{dx}$ in terms of x and y. **[3]**

(ii) Find an expression for y in terms of x, and differentiate it to verify the result in part **(i)**. **[4]**

6 Given that $f(x) = \frac{x + 1}{x - 1}$, show that $ff(x) = x$.

Hence write down the inverse function $f^{-1}(x)$. What can you deduce about the symmetry of the curve $y = f(x)$? **[5]**

7 **(i)** Show that

(*A*) $(x - y)(x^2 + xy + y^2) = x^3 - y^3$,

(*B*) $(x + \frac{1}{2}y)^2 + \frac{3}{4}y^2 = x^2 + xy + y^2$. **[4]**

(ii) Hence prove that, for all real numbers x and y, if $x > y$ then $x^3 > y^3$. **[3]**

Section B (36 marks)

8 Fig. 8 shows the line $y = x$ and parts of the curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$, where

$$\mathrm{f}(x) = \mathrm{e}^{x-1}, \quad \mathrm{g}(x) = 1 + \ln x.$$

The curves intersect the axes at the points A and B, as shown. The curves and the line $y = x$ meet at the point C.

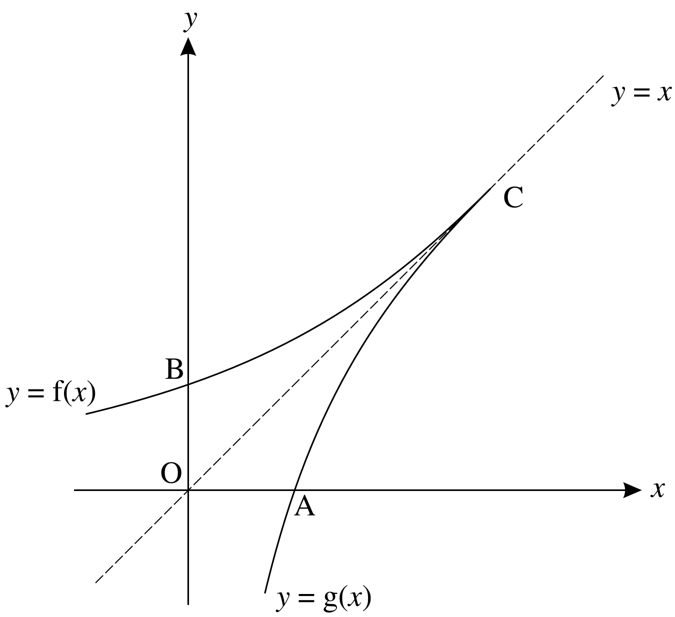


Fig. 8

(i) Find the exact coordinates of A and B. Verify that the coordinates of C are (1, 1). **[5]**

(ii) Prove algebraically that $\mathrm{g}(x)$ is the inverse of $\mathrm{f}(x)$. **[2]**

(iii) Evaluate $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$, giving your answer in terms of e. **[3]**

(iv) Use integration by parts to find $\int \ln x\,\mathrm{d}x$.

Hence show that $\int_{\mathrm{e}^{-1}}^{1} \mathrm{g}(x)\,\mathrm{d}x = \frac{1}{\mathrm{e}}$. **[6]**

(v) Find the area of the region enclosed by the lines OA and OB, and the arcs AC and BC. **[2]**

9 Fig. 9 shows the curve $y = \dfrac{x^2}{3x - 1}$.

P is a turning point, and the curve has a vertical asymptote $x = a$.

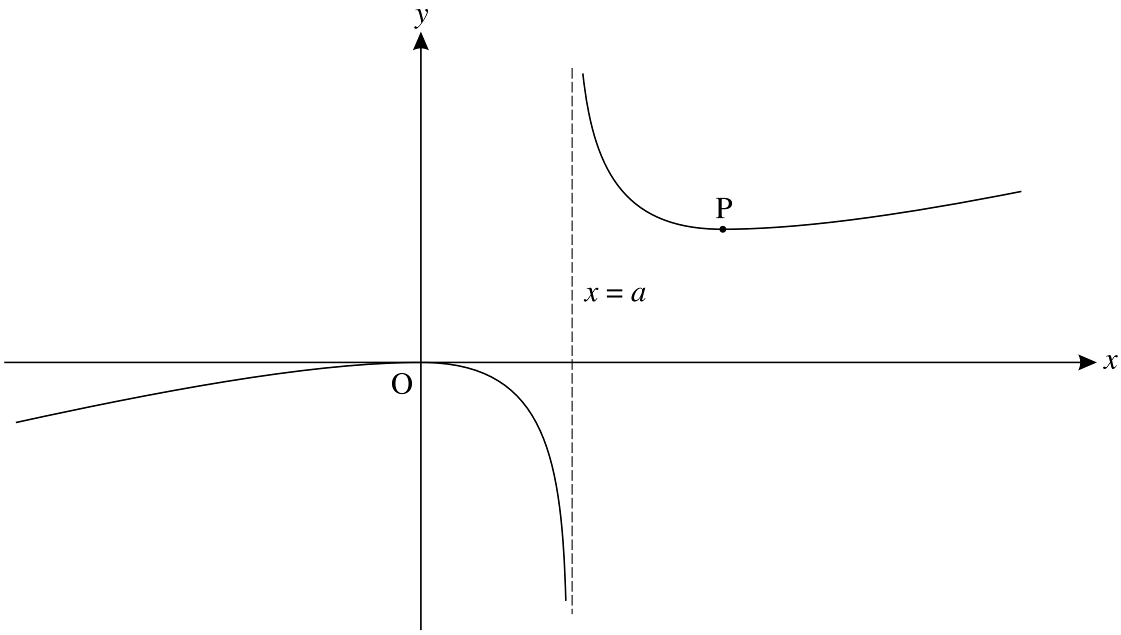


Fig. 9

(i) Write down the value of a. **[1]**

(ii) Show that $\dfrac{dy}{dx} = \dfrac{x(3x-2)}{(3x-1)^2}$. **[3]**

(iii) Find the exact coordinates of the turning point P.

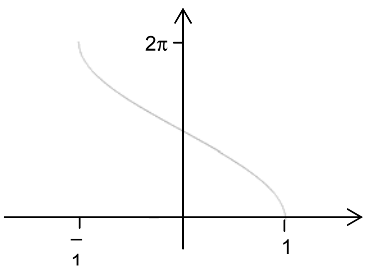
Calculate the gradient of the curve when $x = 0.6$ and $x = 0.8$, and hence verify that P is a minimum point. **[7]**

(iv) Using the substitution $u = 3x - 1$, show that $\displaystyle\int \frac{x^2}{3x-1}\,dx = \frac{1}{27}\int \left(u + 2 + \frac{1}{u}\right) du$.

Hence find the exact area of the region enclosed by the curve, the x-axis and the lines $x = \frac{2}{3}$ and $x = 1$. **[7]**

4753 (C3) Methods for Advanced Mathematics

Section A

1 $\int_0^{\pi/6} \sin 3x \, dx = \left[-\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{6}}$ $= -\frac{1}{3}\cos\frac{\pi}{2} + \frac{1}{3}\cos 0$ $= \frac{1}{3}$	B1 M1 A1cao [3]	$\left[-\frac{1}{3}\cos 3x\right]$ or $\left[-\frac{1}{3}\cos u\right]$ substituting correct limits in $\pm k$ **cos** … 0.33 or better.
2(i) $100 = Ae^0 = A \Rightarrow A = 100$ $50 = 100\,e^{-1500k}$ $\Rightarrow e^{-1500k} = 0.5$ $\Rightarrow -1500k = \ln 0.5$ $\Rightarrow k = -\ln 0.5 / 1500 = 4.62 \times 10^{-4}$	M1A1 M1 M1 A1 [5]	 $50 = A\,e^{-1500k}$ ft their 'A' if used taking lns correctly 0.00046 or better
(ii) $1 = 100e^{-kt}$ $\Rightarrow -kt = \ln 0.01$ $\Rightarrow t = -\ln 0.01 / k$ $= 9966$ years	M1 M1 A1 [3]	ft their A and k taking lns correctly art 9970
3 	M1 B1 A1 [3]	Can use degrees or radians reasonable shape (condone extra range) passes through (–1, 2π), (0, π) and (1, 0) good sketches – look for curve reasonably vertical at (–1, 2π) and (1, 0), negative gradient at (0, π). Domain and range must be clearly marked and correct.
4 $g(x) = 2\lvert x-1\rvert$ $\Rightarrow b = 2\lvert 0-1\rvert = 2$ or (0, 2) $2\lvert x-1\rvert = 0$ $\Rightarrow x = 1$, so $a = 1$ or (1, 0)	 B1 M1 A1 [3]	Allow unsupported answers. www $\lvert x\rvert = 1$ is A0 www

5(i) $e^{2y} = 1 + \sin x$ $\Rightarrow$ $2e^{2y}dy/dx = \cos x$ $\Rightarrow$ $dy/dx = \dfrac{\cos x}{2e^{2y}}$	M1 B1 A1 [3]	Their $2e^{2y} \times dy/dx$ $2e^{2y}$ o.e. cao
(ii) $2y = \ln(1 + \sin x)$ $\Rightarrow$ $y = ½ \ln(1 + \sin x)$ $\Rightarrow$ $dy/dx = \dfrac{1}{2}\dfrac{\cos x}{1+\sin x}$ $= \dfrac{\cos x}{2e^{2y}}$ as before	 B1 M1 B1 E1 [4]	 chain rule (can be within 'correct' quotient rule with $dv/dx = 0$) $1/u$ or $1/(1 + \sin x)$ soi www
6 $\text{f f}(x) = \dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}$ $= \dfrac{x+1+x-1}{x+1-x+1}$ $= 2x/2 = x$* $f^{-1}(x) = f(x)$ Symmetrical about $y = x$.	M1 M1 E1 B1 B1 [5]	correct expression without subsidiary denominators e.g. $= \dfrac{x+1+x-1}{x-1} \times \dfrac{x-1}{x+1-x+1}$ stated, or shown by inverting
7(i) (A) $(x - y)(x^2 + xy + y^2)$ $= x^3 + x^2y + xy^2 - yx^2 - xy^2 - y^3$ $= x^3 - y^3$ * (B) $(x + ½ y)^2 + ¾ y^2$ $= x^2 + xy + ¼ y^2 + ¾ y^2$ $= x^2 + xy + y^2$	M1 E1 M1 E1 [4]	expanding - allow tabulation www $(x + ½ y)^2 = x^2 + ½ xy + ½ xy + ¼ y^2$ o.e. cao www
(ii) $x^3 - y^3 = (x - y)[(x + ½ y)^2 + ¾ y^2]$ $(x + ½ y)^2 + ¾ y^2 > 0$ [as squares ≥ 0] $\Rightarrow$ if $x - y > 0$ then $x^3 - y^3 > 0$ $\Rightarrow$ if $x > y$ then $x^3 > y^3$ *	M1 M1 E1 [3]	substituting results of (i)

8(i) A: $1 + \ln x = 0$ $\Rightarrow$ $\ln x = -1$ so A is $(e^{-1}, 0)$ $\Rightarrow$ $x = e^{-1}$ B: $x = 0$, $y = e^{0-1} = e^{-1}$ so B is $(0, e^{-1})$ C: $f(1) = e^{1-1} = e^0 = 1$ $g(1) = 1 + \ln 1 = 1$	M1 A1 B1 E1 E1 [5]	 SC1 if obtained using symmetry condone use of symmetry Penalise $A = e^{-1}$, $B = e^{-1}$, or co-ords wrong way round, but condone labelling errors.
(ii) *Either* by invertion: e.g. $y = e^{x-1}$ $x \leftrightarrow y$ $x = e^{y-1}$ $\Rightarrow$ $\ln x = y - 1$ $\Rightarrow$ $1 + \ln x = y$	 M1 E1	 taking lns or exps
or by composing e.g. $f\,g(x) = f(1 + \ln x)$ $= e^{1 + \ln x - 1}$ $= e^{\ln x} = x$	 M1 E1 [2]	 $e^{1 + \ln x - 1}$ or $1 + \ln(e^{x-1})$
(iii) $\int_0^1 e^{x-1}dx = \left[e^{x-1}\right]_0^1$ $= e^0 - e^{-1}$ $= 1 - e^{-1}$	M1 M1 A1cao [3]	$\left[e^{x-1}\right]$ o.e or $u = x - 1 \Rightarrow \left[e^u\right]$ substituting correct limits for x or u o.e. not e^0, must be exact.
(iv) $\int \ln x dx = \int \ln x \frac{d}{dx}(x)dx$ $= x\ln x - \int x.\frac{1}{x}dx$ $= x \ln x - x + c$ $\Rightarrow$ $\int_{e^{-1}}^1 g(x)dx = \int_{e^{-1}}^1 (1+\ln x)dx$ $= \left[x + x\ln x - x\right]_{e^{-1}}^1$ $= \left[x\ln x)\right]_{e^{-1}}^1$ $= 1\ln 1 - e^{-1}\ln(e^{-1})$ $= e^{-1}$ *	M1 A1 A1cao B1ft DM1 E1 [6]	parts: $u = \ln x$, $du/dx = 1/x$, $v = x$, $dv/dx = 1$ condone no 'c' ft their '$x \ln x - x$' (provided 'algebraic') substituting limits dep B1 www
(v) Area $= \int_0^1 f(x)\,dx - \int_{e^{-1}}^1 g(x)\,dx$ $= (1 - e^{-1}) - e^{-1}$ $= 1 - 2/e$	M1 A1cao	Must have correct limits 0.264 or better.
or Area OCB = area under curve – triangle $= 1 - e^{-1} - ½ \times 1 \times 1$ $= ½ - e^{-1}$ *or* Area OAC = triangle – area under curve $= ½ \times 1 \times 1 - e^{-1}$ $= ½ - e^{-1}$ Total area $= 2(½ - e^{-1}) = 1 - 2/e$	 M1 A1cao [2]	 OCA or OCB $= ½ - e^{-1}$ 0.264 or better

9(i) $a = 1/3$	B1 [1]	or 0.33 or better
(ii) $\frac{dy}{dx} = \frac{(3x-1)2x - x^2.3}{(3x-1)^2}$ $= \frac{6x^2 - 2x - 3x^2}{(3x-1)^2}$ $= \frac{3x^2 - 2x}{(3x-1)^2}$ $= \frac{x(3x-2)}{(3x-1)^2}$ *	M1 A1 E1 [3]	quotient rule www – must show both steps; penalise missing brackets.
(iii) d*y*/d*x* = 0 when $x(3x-2) = 0$ ⇒ $x = 0$ or $x = 2/3$, so at P, $x = 2/3$ when $x = \frac{2}{3}$, $y = \frac{(2/3)^2}{3\times(2/3)-1} = \frac{4}{9}$ when $x = 0.6$, d*y*/d*x* = –0.1875 when $x = 0.8$, d*y*/d*x* = 0.1633 Gradient increasing ⇒ minimum	M1 A1 M1 A1cao B1 B1 E1 [7]	if denom = 0 also then M0 o.e e.g. $0.\dot{6}$, but must be exact o.e e.g. $0.\dot{4}$, but must be exact –3/16, or –0.19 or better 8/49 or 0.16 or better o.e. e.g. 'from negative to positive'. Allow ft on their gradients, provided –ve and +ve respectively. Accept table with indications of signs of gradient.
(iv) $\int \frac{x^2}{3x-1}dx$ $u = 3x-1 \Rightarrow du = 3dx$ $= \int \frac{\frac{(u+1)^2}{9}}{u}\frac{1}{3}du$ $= \frac{1}{27}\int \frac{(u+1)^2}{u}du = \frac{1}{27}\int \frac{u^2+2u+1}{u}du$ $= \frac{1}{27}\int (u+2+\frac{1}{u})du$ * Area = $\int_{2/3}^{1} \frac{x^2}{3x-1}dx$ When $x = 2/3$, $u = 1$, when $x = 1$, $u = 2$ $= \frac{1}{27}\int_1^2 (u+2+1/u)du$ $= \frac{1}{27}\left[\frac{1}{2}u^2 + 2u + \ln u\right]_1^2$ $= \frac{1}{27}[(2+4+\ln 2) - (\frac{1}{2}+2+\ln 1)]$ $= \frac{1}{27}(3\frac{1}{2}+\ln 2)$ $[= \frac{7+2\ln 2}{54}]$	 B1 M1 M1 E1 B1 M1 A1cao [7]	 $\frac{\frac{(u+1)^2}{9}}{u}$ o.e. × 1/3 (d*u*) expanding Condone missing d*u*'s $\left[\frac{1}{2}u^2 + 2u + \ln u\right]$ substituting correct limits, dep integration o.e., but must evaluate ln 1 = 0 and collect terms.

4753 Methods for Advanced Mathematics (C3) (Written Examination)

General Comments

This paper proved to be accessible to all suitably prepared candidates, and there were plenty of marks available to even the weakest candidates. However, there were some questions, such as 3, 6 and 7(ii) which tested the abler candidates. Fewer full marks than usual were scored – the final part of question 7 proved the main stumbling block – but, equally, virtually all candidates scored above 20. There was no evidence of lack of time to complete the paper.

With reference to recent examiner's reports, it was pleasing to note that fewer candidates were using graph paper for their sketch in question 3. In general, some candidates seem to be insufficiently aware of the significance of words such as 'verify' (see question 8(ii)), and 'hence': most candidates missed the significance of this in questions 6 and 7(ii). Candidates also need to be clear what is meant by exact answers, or else they will lose marks in this paper.

In general, the calculus topics continue to be well answered, albeit with some sloppy notation used in integration, with modulus, proof and inverse trigonometric functions being less securely understood. The standard of presentation varied from chaotic to exemplary.

Comments on Individual Questions

Section A

1) This should have been a routine test of trigonometric integration. However, many candidates confuse differentiation and integration results, using a multiplier of 3 rather than 1/3, and making sign errors. There were also significant numbers of evaluation errors, such as '0' for the lower limit.

2) This question was very well done – exponential growth and decay questions are well understood by the large majority of candidates. The main sources of error lay in the use of '99' instead of '1' in part (ii), and inaccuracy in the final answer through premature rounding of the value of k.

3) Sketching this arccos graph proved to be quite testing – perhaps more so than arcsin - and few candidates scored all three marks. The first M1 was given for a reasonable attempt to reflect a cosine graph in $y = x$. Quite a few candidates scored the 'B1' for showing any graph through (1, 0), (0, π) and (-1, 2π), even if it was a straight line! For the final A1, we wanted to see the correct domain and range, and reasonably correct gradients at $x = -1$, 0 and 1. The use of degrees instead of radians was allowed, as no calculus was involved.

4) Although good candidates did this effortlessly, there was a degree of confusion amongst some candidates over handling the modulus. Many think taking a modulus means you have to multiply by -1, so only did this. The given diagram made most candidates aware that the signs of *a* and *b* were both positive, but there were many incorrect or dubious statements such as "$y = -2$, but y is positive, so $b = 2$" (–2 coming from use of $y = 2x - 2$ when $x = 0$). Another common mistake was to obtain $a = ½$ from the incorrect $2x - 1 = 0$.

Squaring y is another rather dubious technique when applied to a given modulus graph, even more so if the '2' is left un-squared, giving statements like $2(x - 1)^2 = 0$ (true, but from wrong working).

5) (i) Most candidates made a reasonable attempt to differentiate implicitly, but some common errors were (a) starting "d*y*/d*x* = ..", (b) omitting the '2' when differentiating e^{2y}, (c) RHS = 1+ cos *x* , (d) LHS= $2e^{2y\, dy/dx}$.

(ii) The most frequent mistake was in the mishandling of the inversion, with ln 1 + ln sin *x* appearing frequently. Even when the correct expression for *y* was found, a surprising number needlessly used the product or quotient rules with $u = ½$ and $v = \ln(1 + \sin x)$, and/or omitted the derivative of sin *x* in differentiating the latter. Some lost the last mark by not showing clearly that their results in (i) and (ii) were equivalent.

6) This question scored poorly. Although some confused composition with squaring, most candidates managed a correct expression for ff(x) by substituting $(x+1)/(x-1)$ for *x* in f(*x*); however, many of these then failed to deal with the subsidiary denominators, and to correctly simplify the expression to *x*.

Virtually all candidates then tried to invert $y = (x+1)/(x-1)$ to find $f^{-1}(x)$, rather than simply writing down that $f^{-1}(x) = f(x)$. Also, $f^{-1}(x) = (x-1)/(x+1)$ was quite a common error.

There was some confusion in the final B1 between the symmetry of f(*x*) in $y = x$, and the fact that f(*x*) and $f^{-1}(x)$ are symmetrical in this line. Some candidates thought this last question referred to odd and even functions.

7) (i) This algebra proved to be an easy 4 marks for all candidates, give or take a few slips due to carelessness.

(ii) On the other hand, the logic of this part eluded all but the very best candidates. Many substituted values, or tried other letters, or $y = x + 1$, etc. Some recognised that $x^2 + xy + y^2$ had to be proved to be positive, but failed to see the connection between this and (i)(*B*).

Section B

8) Plenty of candidates scored well on this question, seeing the links between the various parts.

(i) Most candidates got the correct coordinates for A and B, although many approximated for e^{-1} – we tried as much as possible to condone this by ignoring subsequent working. It is important that candidates knew why 'verify' was used in finding C – quite a few tried to solve $1 + \ln x = e^{x-1}$.

(ii) Inverting functions is usually well understood, and these two marks were obtained by all but a few candidates.

(iii) Quite a few candidates substituted $u = x - 1$ here, and others left the answer as $e^0 - e^{-1}$ (which though arguably 'in terms of e', was not what was intended), or evaluated e^0 as 0. Some candidates made errors in integrating e^{x-1}, e.g. $(e^{x-1})/(x - 1)$.

(iv) The classic blunder here is to take $u = \ln$ (whatever this means) and $v' = x$! Fortunately, this occurred rarely, and most candidates succeeded in using the correct parts. Quoting this result was not allowed, however, as they were asked to use parts to derive it.

We generously followed through their answer to this to integrate g(x). A rather disconcerting number of candidates – even good ones – failed to simplify $[x + x\ln x - x]$ as $x\ln x$, which made the substitution and derivation of 1/e a bit more complicated than necessary.

(v) This part was more demanding, but a few recovered to achieve the correct answer as the area of the square minus twice the given answer in part (iv).

9) This question was a routine test of calculus which offered plenty of accessible marks.

(i) This was an easy mark for all.

(ii) As this was a routine application of the quotient rule, we withheld the final 'E' mark if the bracket round $(3x - 1)$ was omitted. This was not common, however, and most candidates scored 3 easy marks.

(iii) The coordinates of the turning point were usually obtained correctly from the given derivative, although a few 'burned their boats' by using $3x - 1 = 0$ or numerator = denominator.

The gradients at 0.6 and 0.8 were also well done, and most then explained how this related to P being a minimum point. However, some candidates wasted time and effort in finding the second derivative, usually incorrectly. We allowed this, if fully correct.

(iv) This part proved to be a bit more demanding, and few candidates scored all 7 marks. Errors such as $x = (u - 1)/3$ or $x = (u/3 + 1)$ were found, though most achieved the M1 for substituting du/3 for dx. We condoned missing du's and dx's in this instance, though this is not always the case. Leibnitz would not recognise his notation in some solutions!

The integral was quite often incorrect – usually missing the ln u − and quite a few got the limits for u incorrect, or used the x- limits of 2/3 and 1 instead.